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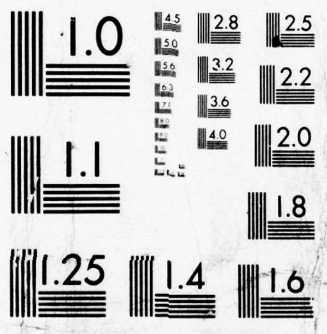
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BY

RAJAT K. DEB

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OPTIMAL DISPATCHING OF A FINITE CAPACITY SHUTTLE

Rajat K. Deb

We consider the problem of determining the optimal operating policy of a two terminal shuttle with fixed capacity $Q \leq \infty$. The passengers arrive at each terminal according to Poisson processes and are transported by a single carrier operating between the terminals. The interterminal travel time is a positive random variable with finite expectation. Under a fairly general cost structure, we show that the policy which minimizes the expected total discounted cost over infinite time horizon has the following form: Suppose the carrier is at one of the terminals with x number of waiting passengers and suppose that y number of passengers are waiting at the other terminal. Then the optimal policy is to dispatch the carrier if and only if $x \geq G(y)$, where $G(y)$ is a monotone decreasing control function. Furthermore, $G(y)$ is always less than or equal to the carrier capacity Q. This control function can be approximated by the linear functions $G(y) = K - \beta y$.

KEY WORDS: Finite Capacity Shuttle, Bulk Queue, Markov Decision Process, Dynamic Programming, Optimal Control, Monotone Policies.

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OPTIMAL DISPATCHING OF A FINITE CAPACITY SHUTTLE

Rajat K. Deb

1. Introduction

We consider the problem of determining optimal control policies for operating a shuttle service between two terminals. The passengers arrive at these terminals (numbered 0 and 1) according to independent Poisson processes $X(t)$ and $Y(t)$ with respective arrival rates λ_0 and λ_1. The carrier that shuttles back and forth between the terminals has a capacity $Q \leq \infty$. The interterminal travel times assumed to be independent random variable with identical distribution $B(\cdot)$ and are independent of everything else. All arriving passengers wait to be transported to the other terminal where they exit the system. The system is reviewed at those points in time when either the carrier has just arrived at one of the terminals, or when the carrier is waiting at one of the terminals and a new passenger arrives. The next arrival may take place at either of the terminals. The state of the system is denoted by (x,y,δ) where x is the number of passengers at terminal 0, y is the number of passengers at terminal 1, and δ is respectively 0 or 1 according to whether the carrier is at terminal 0 or 1 respectively. At each review point and only at these points, one of the following actions is taken: (1) the carrier is dispatched with a batch of passengers (when x customers are at the terminal this batch equals $x \wedge Q \equiv \min\{x,Q\}$), or (2) the carrier is not dispatched. Note that upon taking action (1) or (2) the next control action is taken when the carrier arrives at the other terminal or when next passenger arrives at terminal 0 or 1.

Costs are charged for carrying the passengers and holding the passenger in the system. The cost of carrying y passengers is $R+cy$, where R and c are non-negative constants. The cost of holding x customers is hx per unit time. Without loss of generality we assume that no holding is charged for those passengers who board the carrier before it leaves the terminal.

Our objective is to find a control policy, that is, a sequence of decision rules for selecting actions (1) and (2) at each review point, which minimizes the expected discounted cost over an infinite time horizon. Optimal control policy for the discounted cost case is presented in Section 4 (Theorem 4.2). The optimal policy is of the form: Suppose (x_0, x_1, δ) be the state of the system at a review point. Then the optimal policy is to dispatch the carrier if and only if $x_\delta \geq G_\delta(x_{1-\delta})$, where $G_\delta(x_{1-\delta})$ is a monotone decreasing control function.

There has been relatively little published work on shuttles with stochastic arrivals. Deb and Serfozo [2] determined optimal dispatching rules for a one terminal system. Ignall and Kolesar [4] extended a one terminal system to a two terminal system with infinite capacity, where dispatching decision is made only at one of the terminals. They conjectured that a finite capacity shuttle will have an optimal control rule of the same form. Barnett [1] compares the average number waiting in the system for several control policies. In [6] Ignall et al. have suggested a way of computing the average queue size and the average number of trips for an infinite capacity shuttle under a simple dispatching rule based on the total number of passengers in the system. Our analysis differ from others in the sense that we consider a more realistic finite capacity model and allow the dispatching decisions to be made at both the terminals.

It is interesting to note that the finite capacity shuttle model is also applicable in the cases where a single server attends two queues as in the case of multiplexing and some special cases of multiprogramming. In these cases the server alternates her services between the two queues.

2. Preliminaries

The notation of this section is used throughout this paper. We let $X(t)$ and $Y(t)$ denote the number of arrivals in time t at terminals 0 and 1 respectively. Set $Z(t) = X(t) + Y(t)$ and $\lambda = \lambda_0 + \lambda_1$. Then $Z(t)$ is a Poisson process with intensity λ. Let the random variables τ, ξ_0 and ξ_1 respectively denote an arbitrary interterminal travel time and arbitrary passenger interarrival times at the terminals 0 and 1. The random variables τ, ξ_0 and ξ_1 have respective distributions $1 - \exp(-\lambda_0 t)$, $1 - \exp(\lambda_1 t)$ and $B(t)$. Writing $\xi = \min(\xi_0, \xi_1)$, it can be seen that the random variable ξ has the distribution $1 - \exp(-\lambda t)$. Let $V(x,y,\delta)$ be the optimal α-discounted cost over the infinite time horizon with continuous discounting factor α and initial state (x,y,δ). Note that herein we supress the effect of the discount factor α on V. Since the cost of never dispatching the carrier is

$$(2.1) \qquad E \int_0^\infty \exp(-\alpha t)\ h(x+y+Z(t))dt = h(x+y+\lambda/\alpha)/\alpha < \infty \quad ,$$

the α-discounted cost $V(x,y,\delta) \leq h(x+y+\lambda/\alpha)/\alpha$. Using Theorem 7.1 of [7], which also holds for semi-Markov processes with unbounded costs, we obtain the following optimality equation

$$V(x,y,\delta) = \min\{f(x,y,\delta),\ g(x,y,\delta)\}\ , \tag{2.2}$$

where f is the cost of not dispatching the customer, and holding the waiting customers for a time ξ until the next arrival before taking the next action. $g(x,y,\delta)$ is the cost of dispatching the carrier from the terminal δ, carrying the passengers, holding the excess passengers (if any) and the new arrivals for a period τ, after which another action is taken with the system in state $(x+X(\tau) - (1-\delta)x\wedge Q,\ y+Y(\tau) - \delta(y\wedge Q),\ 1-\delta)$. Clearly,

$$\begin{aligned} f(x,y,\delta) &= E\{\int_0^{\xi} e^{-\alpha t}\, h(x+y)dt + e^{-\alpha\xi}\, V(x+X(\xi),\ y+Y(\xi),\ \delta)\} \\ &= H(x+y) + apV(x+1,y,\delta) + aqV(x,y+1,\delta)\ , \end{aligned} \tag{2.3}$$

where

$$H(x) = E\int_0^{\xi} e^{-\alpha t}\, h(x)dt = hx/(\alpha+\lambda)\ ,$$

$$a = E\{\exp(-\alpha\xi)\} = \lambda/(\alpha+\lambda)\ ,$$

$$p = P[X(\xi) = 1] = P[Y(\xi) = 0] = \lambda_0/(\lambda_0+\lambda_1)\ ,$$

$$q = P[X(\xi) = 0] = P[Y(\xi) = 1] = 1-p\ ,$$

and

$$(2.4)\qquad g(x,y,0) = R + c(x\wedge Q) + E\{\int_0^{\tau} e^{-\alpha t} h(x+y-x\wedge Q+Z(t))dt$$

$$+ e^{-\alpha\tau} V(x-x\wedge Q+X(\tau), y+Y(\tau), 1)\}$$

$$= R + c(x\wedge Q) + \bar{H}(x+y-x\wedge Q) + \sum_{i\geq 0, 0\leq j\leq i} d_{ij} V(x-x\wedge Q+j, y+i-j, 1),$$

$$(2.5)\qquad g(x,y,1) = R + c(y\wedge Q) + \bar{H}(x+y-y\wedge Q)$$

$$+ \sum_{i\geq 0, 0\leq j\leq i} d_{ij} V(x+j, y-y\wedge Q+i-j, 0),$$

where

$$\bar{H}(x) = E\int_0^{\tau} e^{-\alpha t} h(x+Z(t))dt$$

$$= (1-b)hx/\alpha + h\lambda(1-b)/\alpha^2 - h\lambda\alpha^{-1}\int_0^{\infty} te^{-\alpha t}\, dB(t),$$

$$b = E\{\exp(-\alpha\tau)\},\quad d_{ij} = a^i p_i q_{ij},$$

$$p_i = \int_0^{\infty} \frac{\{(\alpha+\lambda)t\}^i}{i!} e^{-(\alpha+\lambda)t}\, dB(t),\quad q_{ij} = \binom{i}{j} p^j q^{i-j}.$$

Note that $\{p_i\}$ and $\{q_{ij}\}$ are probability mass functions, $\Sigma d_{ij} = \Sigma a^i p_i = b$. Whenever both the terms on the right of (2.2) are equal, that is $f = g$, we write

$$(2.6)\qquad V(x,y,\delta) = f(x,y,\delta).$$

Also note that from the definition of H and $\bar{H}$ it follows that

$$(2.7)\qquad H(x) + a\bar{H}(x+1) = \bar{H}(x) + \Sigma d_{ij} H(x+i) = \bar{H}(x) + \sum_{i\geq 0} a^i p_i H(x+i).$$

In addition, for any function $w(x,y,z)$ we define the difference operator Δ as follows

$$(2.8)\qquad \begin{aligned} \Delta w_0(x,y,z) &= w(x,y,z) - w(x-1,y,z_1)\ , \\ \Delta w_1(x,y,z) &= w(x,y,z) - w(x,y-1,z)\ . \end{aligned}$$

For the linear functions H and $\bar{H}$, we define

$$(2.9)\qquad \Delta H = H(x) - H(x-1) = (1-a)h/\alpha \quad \text{and} \quad \Delta\bar{H} = (1-b)h/\alpha\ .$$

We approach the problem of finding an α-optimal using the finite horizon n-period problem which we define in the equation (2.10). We show that for each n and hence in the limit a monotone policy of the form described in Section 1 is optimal. We present these policies in Theorem 4.2. The n-period problem is defined as follows. Let

$$(2.10)\qquad \begin{cases} V^0(x,y,\delta) = h(x+y+\lambda/\alpha)/\alpha\ , \\ V^n(x,y,\delta) = \min\{f^n(x,y,\delta),\ g^n(x,y,\delta)\}\ , \end{cases}$$

where

$$(2.11)\qquad f^n(x,y,\delta) = H(x+y) + apV^{n-1}(x+1,y,\delta) + aqV^{n-1}(x,y+1,\delta)\ ,$$

$$(2.12)\qquad g^n(x,y,0) = R + \bar{H}(x+y-x\wedge Q) + c(x\wedge Q) + \Sigma d_{ij}\ V^{n-1}(x-x\wedge Q+j,\ y+i-j,1),$$

$$(2.13)\qquad g^n(x,y,1) = R + \bar{H}(x+y-y\wedge Q) + c(y\wedge Q) + \Sigma d_{ij}\ V^{n-1}(x+j,\ y-y\wedge Q+i-j,0).$$

The summation on d_{ij} is taken over all $i \geq 0$ and $0 \leq j \leq i$. We can consider V^n as the cost of operating the system for n review periods and incurring a final cost V^0 at the end of the n-th period. Note that if the n-th review takes place at t_n, then $t_n \to \infty$ as $n \to \infty$ and the cost V^0 discounted to time 0 is $\leq E\{\exp(-\alpha t_n)\ h(x+y+Z(t_n))\}$ $= E\{\exp(-\alpha t_n)\ h(x+y+\lambda(t_n))\} \to 0$ as $n \to \infty$. Furthermore, V^n increases monotonically with n, because if $V^n \leq V^{n-1}$ then one can decrease V^{n-1} by using the n-period policy for the (n-1)-period problem. Moreover, from (2.1) we get $h(x+y+\lambda/\alpha)/\alpha \geq V^n(x,y,\delta) > V^{n-1}(x,y,\delta) > 0$ and hence $V^n \uparrow V$ as $n \to \infty$. Now using the Theorem 6.12 of [7] (which also holds for discounted semi-Markov processes) or more appropriately Theorem 2.2 of [9], we assertain that the stationary policy which satisfies equation (2.2) is optimal. Note that in [8,9] the problem is set up for maximization, while we are minimizing the total expected costs. If we take this into account, then it is easy to show that the conditions A1 and A2 of [9] are satisfied and hence the Theorem 2.2 of [9] holds. As a consequence of convergence of V^n to V, we conclude that $f^n \to f$ and $g^n \to g$.

3. n-Period Problem

In this section we consider the n-period problem defined in (2.10) and obtain results similar to those of [3]. Without loss of clarity, we shall often drop the subscripts and arguments of the functions defined in (2.2) - (2.13). For instance, the statement $\Delta V^n > \Delta g^n$ will stand for the statements $\Delta V_0^n(x,y,\delta) > \Delta g_0^n(x,y,\delta)$ and $\Delta V_1^n(x,y,\delta) > \Delta g^n(x,y,\delta)$ for all x, y, and δ.

Lemma 3.1. Let $\Delta f^n \geq \Delta g^n$. Then

(i) $\Delta V^n \leq \Delta f^n$.

(ii) $\Delta V^n \geq \Delta g^n$.

Proof. Clearly, (2.10) implies $V^n \leq f^n$, $V^n \leq g^n$ and for any x, y and δ either

$$(3.1) \qquad V^n(x,y,\delta) = f^n(x,y,\delta)$$

or

$$(3.2) \qquad V^n(x,y,\delta) = g^n(x,y,\delta) \; .$$

If (3.1) holds, then

$$(3.3) \qquad \Delta V_0^n(x+1,y,\delta) \leq \Delta f_0^n(x+1,y,\delta), \quad \Delta V_1^n(x,y+1,\delta) \leq \Delta f_1^n(x,y+1,\delta) \; ,$$

and

$$(3.4) \qquad \Delta V^n(x,y,\delta) \geq \Delta f^n(x,y,\delta) \geq \Delta g^n(x,y,\delta) \; .$$

And if (3.2) holds, then

$$(3.5) \qquad \begin{cases} \Delta V_0^n(x+1,y,\delta) \leq \Delta g_0^n(x+1,y,\delta) \leq \Delta f_0^n(x+1,y,\delta) \; , \\ \Delta V_1^n(x,y+1,\delta) \leq \Delta g_1^n(x,y+1,\delta) \leq \Delta f_1^n(x,y+1,\delta) \; , \end{cases}$$

and

$$(3.6) \qquad \Delta V^n(x,y,\delta) \geq \Delta g^n(x,y,\delta) \; .$$

Now, combining (3.3) and (3.5), we have $\Delta V^n \leq \Delta f^n$. From (3.4) and (3.6) we obtain $\Delta V^n \geq \Delta g^n$.

Lemma 3.2. Let $\Delta f^n \geq \Delta g^n$ and suppose that for all x,y and δ

$$(3.7) \qquad \Delta f_0^n(x,y,\delta) \geq \Delta f_0^n(x,y+1,\delta), \; \Delta f_1^n(x,y,\delta) \geq \Delta f_1^n(x+1,y,\delta) \; ,$$

$$(3.8) \qquad \Delta g_0^n(x,y,\delta) \geq \Delta g_0^n(x,y+1,\delta) \quad \text{and} \quad \Delta g_1^n(x,y,\delta) \geq \Delta g_1^n(x+1,y,\delta) \; .$$

Then

(i) $\Delta V_0^n(x,y,\delta) \geq \Delta V_0^n(x,y+1,\delta)$,

(ii) $\Delta V_1^n(x,y,\delta) \geq \Delta V_1^n(x+1,y,\delta)$.

Proof: The proof is similar to that of Lemma 3.1. Suppose (3.1) holds, then using the fact $V \leq f$, inequality (3.7) and the Lemma 3.1, in that order, we have

$$(3.9)\quad \begin{cases} \Delta V_0^n(x,y,\delta) \geq \Delta f_0^n(x,y,\delta) \geq \Delta f_0^n(x,y+1,\delta) \geq \Delta V_0^n(x,y+1,\delta) , \\ \Delta V_1^n(x,y,\delta) \geq \Delta f_1^n(x,y,\delta) \geq \Delta f_1^n(x+1,y,\delta) \geq \Delta V_1^n(x+1,y,\delta) . \end{cases}$$

And if (3.2) holds then using the fact $V \leq g$, (3.8) and the Lemma 3.1 we obtain

$$(3.10)\quad \begin{cases} \Delta V_0^n(x,y+1,\delta) \leq \Delta g_0^n(x,y+1,\delta) \leq \Delta g_0^n(x,y,\delta) \leq \Delta V_0^n(x,y,\delta) , \\ \Delta V_1^n(x+1,y,\delta) \leq \Delta g_1^n(x+1,y,\delta) \leq \Delta g_1^n(x,y,\delta) \leq \Delta V_1^n(x,y,\delta) . \end{cases}$$

This completes the proof.

Theorem 3.3. If $h > \alpha(c + R/Q)$ then for all x, y and $n \geq 1$

(i) $\Delta f^n \geq \Delta g^n$,

(ii) $\Delta V_0^n(x,y,\delta) \geq \Delta V_0^n(x,y+1,\delta)$,

(iii) $\Delta V_1^n(x,y,\delta) \geq \Delta V_1^n(x+1,y,\delta)$,

(iv) $\Delta V^n \geq c$.

Proof: We prove this Theorem by induction. Using the definition of Δf^n and Δg^n, we have for $n = 1$

$$(3.11) \qquad \Delta f^1 = \Delta H + ah/\alpha = (1-a)h/\alpha + ah/\alpha = h/\alpha .$$

From the definition of $\bar{H}$ and the fact $\Sigma d_{ij} = b$ we have

$$(3.12) \qquad \Delta g_0^1(x,y,1) = \Delta g_1^1(x,y,0) = \Delta\bar{H} + \Sigma d_{ij}\ h/\alpha = h/\alpha ,$$

and

$$(3.13) \qquad \Delta g_\delta^1(x,y,\delta) = \begin{cases} c & \text{for } \delta = 0,\ x \le Q\ (\delta=1,\ y \le Q), \\ \Delta\bar{H} + \Sigma d_{ij}\ h/\alpha = h/\alpha & \text{for } \delta = 0,\ x > Q\ (\delta=1,\ y > Q). \end{cases}$$

Since $h \ge \alpha(c + R/Q)$, therefore (3.11) - (3.13) imply

$$(3.14) \qquad \Delta f^1 \ge \Delta g^1 .$$

Moreover, using (3.11) - (3.13) it is easily varified that the conditions (3.7) and (3.8) of the Lemma 3.2 are satisfied and hence parts (i), (ii) and (iii) of the Theorem 3.3 are true for $n = 1$. To prove part (iv) note that from (3.14), Lemma 3.1, (3.12) and (3.13) we have

$$(3.15) \qquad \Delta V^1 \ge \Delta g^1 \ge \begin{cases} c \\ h/\alpha > c \end{cases} .$$

Now, assuming that the Theorem is true for all $n \le k$, then for $n = k+1$, we have

$$
\begin{aligned}
(3.16)\qquad \Delta g_1^{k+1}(x,y,0) &= \Delta\bar{H} + \Sigma d_{ij}\ \Delta V_1^k(x-x\Lambda Q+j,\ y+i-j,\ 1) \\
&\le \Delta\bar{H} + \Sigma d_{ij}\ \Delta f_1^k(x-x\Lambda Q+j,\ y+i-j,\ 1) \\
&= \Delta\bar{H} + \Sigma d_{ij}\{\Delta H + ap\ \Delta V_1^{k-1}(x-x\Lambda Q+j+1,\ y+i-j,\ 1) \\
&\qquad + aq\ \Delta V_1^{k-1}(x-x\Lambda Q+j,\ y+1+i-j,\ 1)\} \\
&\le \Delta\bar{H} + \Sigma d_{ij}\ \Delta H + a\Sigma d_{ij}\{p\Delta V_1^{k-1}(x+1-(x+1)\Lambda Q+j,\ y+i-j,\ 1) \\
&\qquad + q\Delta V_1^{k-1}(x-x\Lambda Q+j,\ y+1+i-j,\ 1)\} \\
&= \Delta H + ap\{\Delta\bar{H} + \Sigma d_{ij}\ \Delta V_1^{k-1}(x+1-(x+1)\Lambda Q+j,\ y+i-j,\ 1)\} \\
&\qquad + aq\{\Delta\bar{H} + \Sigma d_{ij}\ \Delta V_1^{k-1}(x-x\Lambda Q+j,\ y+i-j+1,\ 1\} \\
&= \Delta H + ap\ \Delta g_1^k(x+1,y,0) + aq\ \Delta g_1^k(x,y+1,0) \\
&\le \Delta H + ap\ \Delta V_1^k(x+1,y,0) + aq\ \Delta V_1^k(x,y+1,0) \\
&= \Delta f_1^{k+1}(x,y,0)\ .
\end{aligned}
$$

The lines 1, 3, 6 and 8 of (3.16) follows from the definitions of Δg and Δf. Lines 2 and 7 follows from the fact that $\Delta V^k \le \Delta f^k$ and $\Delta g^k \le \Delta V^k$. Also note that for $x \ge Q$, $x-x\Lambda Q+1 = (x+1) - (x+1)\Lambda Q$ and for $x \le Q-1$, $x-x\Lambda Q = (x+1) - (x+1)\Lambda Q$. Therefore, for $x \le Q-1$ line 4 equals line 3 and for $x \ge Q$, we obtain line 4 by using the induction hypothesis $\Delta V_1^{k-1}(x+1,y,1) \le \Delta V_1^{k-1}(x,y,1)$ for all x and y. Finally, line 5 can be obtained from (2.7) by noting that $p+q = 1$. Using symmetry and arguments similar to those used in (3.16) we obtain

(3.17) $$\Delta g_0^{k+1}(x,y,1) \le \Delta f_0^{k+1}(x,y,1) \quad .$$

Moreover, from (2.9) and the induction hypothesis $\Delta V^k \ge c$ we have for $x \le Q$

(3.18) $$\begin{aligned}\Delta g_0^{k+1}(x,y,0) &= c = (1-a)c + ac < (1-a)h/\alpha + c \\ &= \Delta H + apc + aqc \le \Delta H + ap\,\Delta V_0^k(x+1,y,0) + aq\,\Delta V^k(x,y+1,0) \\ &= \Delta f_0^{k+1}(x,y,0) \quad .\end{aligned}$$

For $x > Q$, using the fact that $\Delta V^k \le \Delta f^k$, $\Delta g^k \le \Delta V^k$, definitions of Δf and Δg, and the quality (2.7) we have

(3.19) $$\begin{aligned}\Delta g_0^{k+1}(x,y,0) &= \Delta\bar{H} + \Sigma d_{ij}\,\Delta V_0^k(x-Q+j,\ y+i-j,\ 1) \\ &\le \Delta\bar{H} + \Sigma d_{ij}\,\Delta f_0^k(x-Q+j,\ y+i-j,\ 1) \\ &= \Delta\bar{H} + \Sigma d_{ij}\,\Delta H + \Sigma d_{ij}\{ap\,\Delta V_0^{k-1}(x-Q+1+j,\ y+i-j,\ 1) \\ &\qquad + aq\,\Delta V_0^{k-1}(x-Q+j,\ y+i-j+1,\ 1)\} \\ &= \Delta H + ap\{\Delta\bar{H} + \Sigma d_{ij}\,\Delta V_0^{k-1}(x-Q+1+j,\ y+i-j,\ 1)\} \\ &\qquad + aq\{\Delta\bar{H} + \Sigma d_{ij}\,\Delta V_0^{k-1}(x-Q+j,\ y+1+i-j,\ 1)\} \\ &= \Delta H + ap\,\Delta g_0^k(x+1,y,0) + aq\,\Delta g_0^k(x,y+1,0) \\ &\le \Delta H + ap\,\Delta V_0^k(x+1,y,0) + aq\,\Delta V_0^k(x,y+1,0) \\ &= \Delta f_0^{k+1}(x,y,0) \quad .\end{aligned}$$

Combining (3.18) and (3.19) we obtain $\Delta g_0^{k+1}(x,y,0) \leq \Delta f_0^{k+1}(x,y,0)$ using arguments similar to those used in (3.18) and (3.19) we can show that $\Delta g_1^{k+1}(x,y,1) \leq \Delta f_1^{k+1}(x,y,1)$ and hence

$$\Delta g^{k+1} \leq \Delta f^{k+1} \quad . \tag{3.20}$$

To prove parts (ii) and (iii) of this theorem it is sufficient to show that the condition (3.7) and (3.8) of the Lemma 3.2 are satisfied. Using the induction hypothesis and definition of Δf we have

$$\begin{aligned}\Delta f_0^{k+1}(x,y+1,\delta) &= \Delta H + ap\,\Delta V_0^k(x+1,y+1,\delta) - aq\,\Delta V_0^k(x,y+2,\delta)\\ &\leq \Delta H + ap\,\Delta V_0^k(x+1,y,\delta) + aq\,\Delta V_0^k(x,y+1,\delta)\\ &= \Delta f_0^K(x,y,\delta) \quad ,\end{aligned}$$

and

$$\begin{aligned}\Delta f_1^{k+1}(x+1,y,\delta) &= \Delta H + ap\,\Delta V_1^k(x+2,y,\delta) + aq\,\Delta V_1^k(x+1,y+1,\delta)\\ &\leq \Delta H + ap\,\Delta V_1^k(x+1,y,\delta) + aq\,\Delta V_1^k(x,y+1,\delta)\\ &= \Delta f_1^k(x,y,\delta) \quad .\end{aligned}$$

To prove (3.8), note that for $\delta = 0$, $x < Q$, $y \geq 0$

$$\Delta g_0^{k+1}(x,y,\delta) = c \ , \tag{3.21}$$

and for $\delta = 0$, $x \geq Q$, $y \geq 0$ and $\delta = 1$, $x \geq 0$, $y \geq 0$, using the fact $\Delta V_0^k(x,y+1,\delta) \leq \Delta V_0^k(x,y,\delta)$ for all x, y and δ, we have

$$(3.22) \quad \begin{aligned} \Delta g_0^{k+1}(x,y+1,\delta) &= \Delta\bar{H} + \Sigma d_{ij}\ V_0^k(x-(1-\delta)Q+j,\ y+1-\delta(y+1)\wedge Q+i-j,\ 1-\delta) \\ &\leq \Delta\bar{H} + \Sigma d_{ij}\ V_0^k(x-(1-\delta)Q+j,\ y-\delta(y\wedge Q)+i-j,\ 1-\delta) \\ &= \Delta g_0^{k+1}(x,y,\delta) \quad . \end{aligned}$$

Therefore, using (3.21) and (3.22) we obtain

$$(3.23) \qquad \Delta g_0^{k+1}(x,y,\delta) \geq \Delta g_0^{k+1}(x,y+1,\delta) \quad .$$

Using arguments similar to those used in (3.23) we obtain

$$(3.24) \qquad \Delta g_1^{k+1}(x,y,\delta) \geq \Delta g_1^{k+1}(x+1,y,\delta) \quad .$$

Finally to prove part (iv) of this theorem, we note that (3.20) implies that the Lemma 3.1 is true for $n = k+1$ and hence

$$(3.26) \quad \begin{aligned} \Delta V^{k+1}(x,y,\delta) &\geq \Delta g^{k+1}(x,y,\delta) \\ &= c \qquad \text{for } \delta = 0,\ x \leq Q\ (\delta = 1,\ y \leq Q)\ , \\ &= \Delta\bar{H} + \Sigma d_{ij}\ \Delta V^k(x-(1-\delta)Q+j,\ y-\delta Q+i-j,\ 1-\delta) \qquad \text{otherwise} \\ &\geq \Delta\bar{H} + \Sigma d_{ij} c \geq c\ . \end{aligned}$$

The last line in the above is obtained from the induction hypothesis and the fact that $h/\alpha > c$. This completes the proof.

Lemma 3.4. Let $h > \alpha(c + R/Q)$. Then

(i) $V^n(x,y,0) = g^n(x,y,0)$ for all $x \geq Q$,

(ii) $V^n(x,y,1) = g^n(x,y,1)$ for all $y \geq Q$.

Proof. From (2.10) - (2.13) we have

$$(3.26) \qquad f^1(x,y,\delta) = H(x+y) + ah(x+y+1 + \frac{\lambda}{\alpha})/\alpha$$

$$= h(x+y+ \lambda/\alpha)/\alpha \quad ,$$

and from $\delta = 0$, writing $e = x \wedge Q$ we have

$$(3.27) \qquad g^1(x,y,0) = R + ce + \bar{H}(x+y-e) + \Sigma d_{ij}(x+y-e+i+\lambda/\alpha)h/\alpha \ .$$

From definition of $\bar{H}(x)$ we have

$$(3.28) \qquad \bar{H}(x) = (1-b)hx/\alpha + \{(1-b)/\alpha - \int_0^\infty t \exp(-\alpha t)\, dB(t)\}\, h\lambda/\alpha$$

$$= (1-b)hx/\alpha + \{\lambda(1-b)/\alpha - \sum n\, a^n\, p_n\}\, h/\alpha$$

$$= (1-b)hx/\alpha + \{\lambda(1-b)/\alpha - \sum id_{ij}\}\, h/\alpha \quad .$$

Using (3.27), (3.28) and noting that $\Sigma d_{ij} = b$, we have

$$(3.29) \qquad g^1(x,y,0) = (R + ce - he/\alpha) + (x+y+\lambda/\alpha)\, h/\alpha \quad .$$

If $x \geq Q$, then $e = Q$, and using the fact $h > \alpha(c + R/Q)$, we have $g^1(x,y,0) < f^1(x,y,0)$. Hence $V^1(x,y,0) = g^1(x,y,0)$. Now, suppose that for all $x \geq Q$ and $n \leq k$, $f^n(x,y,0) > g^n(x,y,0)$ and consequently, $V^n(x,y,0) = g^n(x,y,0)$. Then using the induction hypothesis, equation (2.7) and the fact $h > \alpha(c + R/Q)$, we have

$$
\begin{aligned}
f^{k+1}(x,y,0) &= H(x+y) + ap\, V^k(x+1,y,0) + aq\, V^k(x,y+1,0) \\
&= H(x+y) + ap\{R + \bar{H}(x+y+1-Q) + cQ \\
&\qquad\qquad + \Sigma d_{ij}\, V^{k-1}(x+1-Q+j,\ y+i-j,\ 1)\} \\
&\qquad + aq\{R + \bar{H}(x+y+1-Q) + cQ \\
&\qquad\qquad + \Sigma d_{ij}\, V^{k-1}(x-Q+j,\ y+1+i-j,\ 1)\} \\
&= H(Q) + \bar{H}(x+y-Q) + a(R+cQ) + \Sigma d_{ij}\{H(x+y+1-Q+i) \\
&\qquad + ap\, V^{k-1}(x+1-Q+j,\ y+i-j,\ 1) \\
&\qquad + aq\, V^{k-1}(x-Q+j,\ y+1+i-j,\ 1)\} \\
&\geq H(Q) - (1-a)\,(R+cQ) + R + cQ + \bar{H}(x+y-Q) \\
&\qquad\qquad + \Sigma d_{ij}\, V^k(x-Q+j,\ y+i-j,\ 1) \\
&= (1-a)\, Q\{h/\alpha - c - R/Q\} + g^{k+1}(x,y,0) \\
&> g^{k+1}(x,y,0)\ .
\end{aligned}
$$

For $\delta = 1$ and $y \geq Q$, the proof is analogous.

<u>Lemma 3.5</u>. Let $h \leq \alpha(c + R/Q)$. Then for all $x \geq 0$, $y \geq 0$ and $\delta \in \{0,1\}$

$$V^n(x,y,\delta) = f^n(x,y,\delta) = (x+y+\lambda/\alpha)\ h/\alpha\ .$$

Proof. Using (3.26) - (3.29) and letting $e = x$ for $\delta = 0$ and $e = y$ for $\delta = 1$, we have

$$f^1(x,y,\delta) = (x+y+\lambda/\alpha)\ h/\alpha$$

and

$$g^1(x,y,\delta) = R + c(e\wedge Q) - h(e\wedge Q)/\alpha + (x+y+\lambda/\alpha)\ h/\alpha\ .$$

Note that $h(e\wedge Q) \leq c(e\wedge Q) + R(e\wedge Q)/Q \leq c(e\wedge Q)+R$. Hence $g^1(x,y,0) \geq f^1(x,y,0) = (x+y+\lambda/\alpha)h/\alpha = V^1(x,y,0)$. Now, suppose that the Lemma is true for all $n \leq k$. Then for $n = k+1$,

$$\begin{aligned} g^{k+1}(x,y,0) &= R + c(x\wedge Q) + \bar{H}(x+y-x\wedge Q) + \Sigma d_{ij}\ V^k(x-x\wedge Q+j,\ y+i-j,\ 1) \\ &= R + c(x\wedge Q) + \bar{H}(x+y-x\wedge Q) + \Sigma d_{ij}(x+y-x\wedge Q+i+\lambda/\alpha)h/\alpha \\ &\geq (x+y+\lambda/\alpha)h/\alpha = f^{k+1}(x,y,0)\ . \end{aligned}$$

Similarly one can show that $g^{k+1}(x,y,1) \geq f^{k+1}(x,y,1) = (x+y+\lambda/\alpha)\ h/\alpha$. Hence $V^n(x,y,\delta) = f^n(x,y,\delta) = (x+y+\lambda/\alpha)\ h/\alpha$ for all $n \geq 1$.

4. Optimal Discounted Cost Policy

In this section we present the main results of this paper. These results are obtained directly from the lemmas and the Theorem of the Section 3. We summarize these results in the following corollary.

Corollary 4.1.

(i) If $h \leq a(c + R/Q)$, then for all x, y and δ

$$V(x,y,\delta) = f(x,y,\delta) = (x+y+\lambda/\alpha)\, h/\alpha \quad .$$

(ii) If $h > \alpha(c + R/Q)$, then

(a) $\Delta f \geq \Delta g$, $\Delta V \leq \Delta f$, $\Delta V \geq \Delta g$;

(b) $\Delta V_0(x,y,\delta) \geq \Delta V_0(x,y+1,\delta)$, $\Delta V_1(x,y,\delta) \geq \Delta V_1(x+1,y,\delta)$;

(c) $\Delta V \geq c$;

(d) For all $x \geq Q$, $y \geq 0$ and $\delta = 0$ $(x \geq 0,\ y \geq Q,\ \delta = 1)$

$V(x,y,\delta) = g(x,y,\delta)$.

From the comments at the end of Section 2, we know that $V^n \to V$, $g^n \to g$ and $f^n \to f$, and hence $\Delta V^n \to \Delta V$. Therefore, 4.1(i) follows from the Lemma 3.5. Parts (a), (b) and (c) of the Corollary 4.1(ii) come from Theorem 3.3 and Lemma 3.1. Finally, part (d) in the above is obtained from Lemma 3.4.

Theorem 4.2.

(i) If $h \leq \alpha(c + R/Q)$, then the policy of never dispatching the carrier is α-optimal and $V(x,y,\delta) = h(x+y+\lambda/\alpha)/\alpha$.

(ii) Let $h > \alpha(c + R/Q)$ and (x,y,δ) be the state of the system. Then for $\delta = 0$ $(\delta = 1)$, there is a critical number $G_0(y) \leq Q$ $(G_1(x) \leq Q)$, such that following is an optimal policy:

Dispatch the carrier if $x \geq G_0(y)$ $(y \geq G_1(x))$ and wait for the next arrival if $x < G_0(y)$ $(y < G_1(x))$.

Furthermore, $G_0(y)$ and $G_1(y)$ are respectively monotone decreasing functions of y and x.

<u>Proof</u>. The first part of this Theorem follows directly from the Corollary 4.1(i) and the fact that $f(x,y,\delta)$ is the cost of the action of not disptaching the carrier.

Now, we prove the second part of the theorem for $\delta = 0$. The proof for the case $\delta = 1$ is similar to the case $\delta = 0$. Let y be fixed and

$$G_0(y) = \min\{x \geq 0 : f(x,y,0) > g(x,y,0)\} . \tag{4.1}$$

From the Corollary 4.1(ii)(d), it follows that for all $x \geq Q$ $f(x,y,0) > g(x,y,0) = V(x,y,0)$ and hence $G_0(y) \leq Q$. Now, it suffices to show that

$$V(x,y,0) = g(x,y,0) \qquad \text{for all } x \geq G_0(y) . \tag{4.2}$$

We establish the validity of (4.2) by induction. Clearly (4.2) holds for $x = G_0(y)$. Assume that (4.2) is true for some $x \geq G_0(y)$. Then using the Corollary 4.1(ii)(a), we have

$$V(x+1,y,0) - V(x,y,0) \geq g(x+1,y,0) - g(x,y,0) . \tag{4.3}$$

Then using (4.3) and the induction hypothesis, which imply $V(x,y,0) = g(x,y,0)$, we obtain $V(x+1,y,0) \geq g(x+1,y,0)$. But from (2.2). We know $V(x,y,0) \leq g(x,y,0)$ and hence $V(x+1,y,0) = g(x+1,y,0)$. Again from the definition (4.1) of $G_0(y)$, it follows that $V(x,y,0) = f(x,y,0)$ for all $x < G_0(y)$. To show $G_0(y)$ is monotonically decreasing, we note that due to the Corollary 4.1(ii)(a), we have

$$f(x,y+1,0) - f(x,y,0) \geq g(x,y+1,0) - g(x,y,0) \tag{4.4}$$

and from (4.1) we get

$$f(x,y,0) > g(x,y,0) \qquad \text{for all } x \geq G_0(y)\ . \tag{4.5}$$

Combining (4.4) and (4.5), we get $f(x,y+1,0) > g(x,y+1,0)$ and hence $G_0(y+1) \leq G_0(y)$. This completes the proof.

Since the control function G is non-increasing, we can approximate G by a linear function. The family of monotone decreasing linear functions have the form

$$G_0(y) = K_0-\beta_0 y \quad \text{and} \quad G_1(x) = K_1-\beta_1 x \quad \text{where} \quad 0 \leq \beta_i \leq K_i,\ i = 0,\ 1.$$

In this case the approximately optimal policy is completely determined by the critical numbers $(K_0, \beta_0, K_1, \beta_1)$. Such simple policies are ideally suited for practical applications. The values of these critical constants can be determined by standard methods of calculus [2,3]. Some methods for computing these constants for $\beta_0 = \beta_1 = 1$ can also be found in [6]. For small Q, one can obtain the values of G by direct enumeration, because $G \leq Q$. We hope to present some exact and approximate methods for computing G in the near future.

Finally, one can show that the Theorem 4.2 is valid for $Q = +\infty$. However, in this case the Lemma 3.4 and Corollary 4.1(ii)(d) are invalid. The fact that $G < \infty$ can be established by the method given in the Lemma 3.2 of [2]. One can also readily extend these results for non-linear holding costs.

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20. Abstract

OPTIMAL DISPATCHING OF A FINITE CAPACITY SHUTTLE

< or = infinity

Rajat K. Deb

We consider the problem of determining the optimal operating policy of a two terminal shuttle with fixed capacity $Q \leq \infty$. The passengers arrive at each terminal according to Poisson processes and are transported by a single carrier operating between the terminals. The interterminal travel time is a positive random variable with finite expectation. Under a fairly general cost structure, it is shown ~~we show~~ that the policy which minimizes the expected total discounted cost over infinite time horizon has the following form: Suppose the carrier is at one of the terminals with x number of waiting passengers and suppose that y number of passengers are waiting at the other terminal. Then the optimal policy is to dispatch the carrier if and only if $x \geq G(y)$, where $G(y)$ is a monotone decreasing control function. Furthermore, $G(y)$ is always less than or equal to the carrier capacity Q. This control function can be approximated by the linear functions $G(y) = K - \beta y$.

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